Stability Analysis of Networked Control Systems

A Sum of Squares Approach

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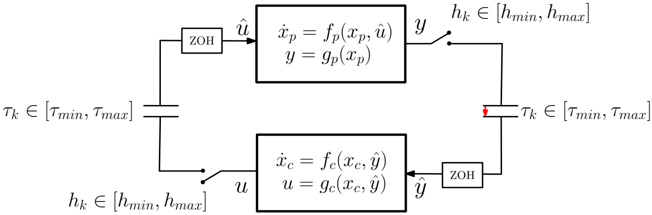
Where innovation starts

NCS Model



Network effects:

- varying transmission intervals $h_k \in [h_{min}, h_{max}], \forall k$
- varying delays $\tau_k \in [\tau_{min}, \tau_{max}]$, where $\tau_k \leq \min\{\tau_{max}, h_k\}, \forall k$
- communication constraints

$\rightarrow$ Stability Analysis

Communication Constraints

Node:	$\sigma \in \{1, ..., N\}$	a collection of sensors and/or actuators that can transmit data simultaneously
Protocol:	$\delta(\sigma, e)$	a function that determines which node is allowed to transmit data
Network Error:	$e \in \mathbb{R}^{n_e}$	$e = [e_u^\top, e_y^\top]^\top$ where $e_u = \hat{u} - u$ and $e_y = \hat{y} - y$

RR protocol:

$$\sigma_{NEW} = \delta_{RR}(\sigma_{OLD}, e) := (\sigma \ mod \ N) + 1$$

TOD/MEF protocol:

$$\sigma_{NEW} = \delta_{TOD}(\sigma_{OLD}, e) := \arg\max |e_j|, \quad j = 1, ..., N$$

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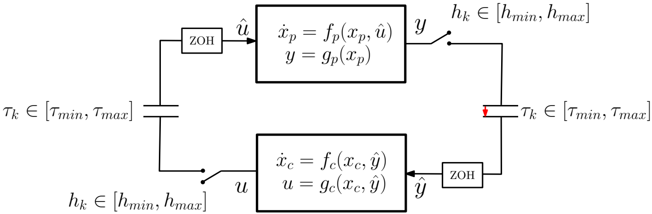
NCS Model



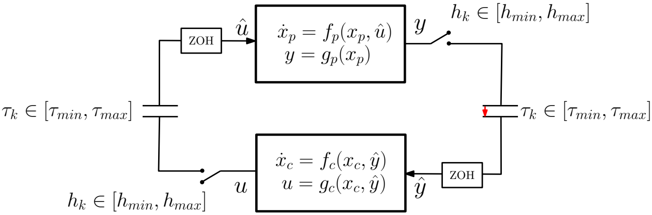
At updates at times $t_k + \tau_k$, $k \in \mathbb{N}$

$$\hat{y}((t_k + \tau_k)^+) = y(t_k) + \tilde{h}(\delta(\sigma(t_k), e(t_k)), e(t_k))$$

where

$$\tilde{h}_i(\delta(\sigma(t_k), e(t_k)), e(t_k)) := \begin{cases} 0, & \text{if } i = \delta(\sigma(t_k), e(t_k)), \\ e_i(t_k), & \text{otherwise} \end{cases}$$

NCS Model



$$\begin{aligned} \dot{\xi} &= F(\xi), \quad \xi \in C, \\ \xi^+ &= G(\xi), \quad \xi \in D \end{aligned}$$

[1] Goebel, Sanfelice, and Teel. *IEEE Control Systems Magazine, April 2009.*

[2] Nešić and Teel. *IEEE Transactions on Automatic Control, October 2004.*

[3] Heemels et al. *IEEE Transactions on Automatic Control, August 2010.*

Hybrid Model

State variables:

- $x \in \mathbb{R}^{n_x}$ - combined state $[x_p^\top, x_c^\top]^\top$ of the plant and controller
- $e \in \mathbb{R}^{n_e}$ - the network-induced error $[e_u^\top, e_y^\top]^\top$
- $s \in \mathbb{R}^{n_e}$ - memory storing the value $h(\sigma, e) - e$ at t_k
- $\tau \in \mathbb{R}_{\geq 0}$ - timer to guarantee $\tau_k \in [\tau_{min}, \tau_{max}]$, $h_k \in [h_{min}, h_{max}]$
- $\sigma \in \{1, ..., N\}$ - latest node that got access to the network
- $\ell \in \{0, 1\}$ - signifies a transmission or an update

the entire state: $\xi = [x^\top, e^\top, s^\top, \tau, \sigma, \ell]^\top \in \mathbb{R}^{n_\xi}$

The hybrid system model

$$\begin{aligned} \dot{\xi} &= F(\xi), \quad \xi \in C, \\ \xi^+ &= G(\xi), \quad \xi \in D \end{aligned}$$

where

$$C := \{\xi \in \mathbb{R}^{n_\xi} \mid (\ell = 0 \wedge \tau \in [0, h_{max}]) \vee (\ell = 1 \wedge \tau \in [0, \tau_{max}])\}$$

$$F(\xi) = F(x, e, s, \tau, \sigma, \ell) := (f(x, e), g(x, e), 0, 1, 0, 0)$$

and

$$D := \{\xi \in \mathbb{R}^{n_\xi} \mid (\ell = 0 \wedge \tau \in [h_{min}, h_{max}]) \vee (\ell = 1 \wedge \tau \in [\tau_{min}, \tau_{max}])\}$$

$$G(\xi) := \begin{cases} G(x, e, s, \tau, \sigma, 0) = (x, e, \tilde{h}(\delta(\sigma, e), e) - e, 0, \delta(\sigma, e), 1) & \{t_k\} \\ G(x, e, s, \tau, \sigma, 1) = (x, s - e, 0, \tau, \sigma, 0) & \{t_k + \tau_k\} \end{cases}$$

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Hybrid Stability via SOS

THEOREM: Consider a hybrid system $\Sigma = (C, F, D, G)$ and a compact set $\mathcal{A} \subset \mathbb{R}^{n_\xi}$ satisfying $G(D \cap \mathcal{A}) \subset \mathcal{A}$. If every solution of Σ exists for all times $t \in [0, \infty)$ and there exists a Lyapunov function candidate V for $(\Sigma, \mathcal{A})$ that satisfies

$$\langle \nabla V(\xi), F(\xi) \rangle < 0 \text{ for all } \xi \in C \backslash \mathcal{A}$$
$$V(G(\xi)) - V(\xi) \leq 0 \text{ for all } \xi \in D \backslash \mathcal{A},$$

then the set $\mathcal{A}$ is GAS.

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DEFINITION: A polynomial $p(x)$ is called a sum of squares (SOS) if there exist polynomials $p_1(x), p_2(x), \dots, p_m(x)$ such that $p(x) = \sum_{i=1}^{m} p_i^2(x)$ for all $x \in \mathbb{R}^n$.

- SOS can be checked via LMI conditions!!

Piecewise Polynomial Hybrid System

$$\begin{aligned} \dot{\xi} &= F(\xi), \quad \xi \in C, \\ \xi^+ &= G(\xi), \quad \xi \in D \end{aligned}$$

↓

$$\begin{aligned} \dot{\xi} &= F_i(\xi), \quad \xi \in C_i, \\ \xi^+ &= G_m(\xi), \quad \xi \in D_m \end{aligned}$$

$$\begin{aligned} C_i &= \{\xi \in \mathbb{R}^{n_\xi} \mid c_{i,j}(\xi) \geq 0, \text{ for } j = 1, .., m_C^i, \\ &\qquad \bar{c}_{i,l}(\xi) = 0, \text{ for } l = 1, .., n_C^i\}, \\ D_m &= \{\xi \in \mathbb{R}^{n_\xi} \mid d_{m,j}(\xi) \geq 0, \text{ for } j = 1, .., m_D^m, \\ &\qquad \bar{d}_{m,l}(\xi) = 0, \text{ for } l = 1, .., n_D^m\} \end{aligned}$$

- basic semialgebraic sets

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NCS Stability via SOS

- Is $F(\xi)$ (piecewise) polynomial?

$$F(\xi) = F(x, e, s, \tau, \sigma, \ell) := (f(x, e), g(x, e), 0, 1, 0, 0)$$

$$C \;:=\; \{\xi \in \mathbb{R}^{n_\xi} \mid (\ell = 0 \wedge \tau \in [0, h_{max}]) \vee (\ell = 1 \wedge \tau \in [0, \tau_{max}])\}$$

- Yes! ✓

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- Is $G(\xi)$ (piecewise) polynomial?

$$G(\xi) := \begin{cases} G(x, e, s, \tau, \sigma, 0) & = (x, e, \tilde{h}(\delta(\sigma, e), e) - e, 0, \delta(\sigma, e), 1) \\ G(x, e, s, \tau, \sigma, 1) & = (x, s - e, 0, \tau, \sigma, 0) \end{cases}$$

$$D \ := \ \{\xi \in \mathbb{R}^{n_\xi} \mid (\ell = 0 \wedge \tau \in [h_{min}, h_{max}]) \vee (\ell = 1 \wedge \tau \in [\tau_{min}, \tau_{max}])\}$$

- Doesn't appear so ...

IDEA: Encode the protocol $\delta(\sigma, e)$ into the jump set D

$G(\xi)$ is piecewise polynomial ...

$$\begin{aligned} G(x, e, s, \tau, j, 0) &= (x, e, \tilde{h}(j, e) - e, 0, j, 1), \quad j = 1, \ldots, N \\ G(x, e, s, \tau, \sigma, 1) &= (x, s + e, 0, \tau, \sigma, 0) \end{aligned}$$

for $j = 1, \ldots, N$, corresponding jump map

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for $j = 1, \ldots, N$, corresponding jump map

$$RR \begin{cases} D_{0,1} = \{ \xi \in \mathbb{R}^{n_\xi} \mid \ell = 0, \ \tau - h_{min} \geq 0, h_{max} - \tau \geq 0, \sigma = N \} \\ D_{0,j} = \{ \xi \in \mathbb{R}^{n_\xi} \mid \ell = 0, \ \tau - h_{min} \geq 0, h_{max} - \tau \geq 0, \sigma + 1 = j \} \end{cases}$$

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$$
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G(x, e, s, \tau, \sigma, 1) &= (x, s + e, 0, \tau, \sigma, 0)
\end{aligned}
$$

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TOD \begin{cases} D_{0,j} = \{ \xi \in \mathbb{R}^{n_\xi} \mid \ell = 0,\ \tau - h_{min} \geq 0, h_{max} - \tau \geq 0, \\ \qquad e_j^\top e_j - e_i^\top e_i \geq 0\ \forall i = 1, \ldots, N \} \end{cases}
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NCS Stability via SOS

IDEA: Encode the protocol $\delta(\sigma, e)$ into the jump set D

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$$G(x, e, s, \tau, j, 0) = (x, e, \tilde{h}(j, e) - e, 0, j, 1), \quad j = 1, \dots, N$$
$$G(x, e, s, \tau, \sigma, 1) = (x, s + e, 0, \tau, \sigma, 0)$$

for $j = 1, \dots, N$, corresponding jump map

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$$TOD \begin{cases} D_{0,j} = \{ \xi \in \mathbb{R}^{n_\xi} \mid \ell = 0,\ \tau - h_{min} \geq 0, h_{max} - \tau \geq 0, \\ \qquad e_j^\top e_j - e_i^\top e_i \geq 0\ \forall i = 1, \dots, N\} \end{cases}$$

with

$$D_1 = \{ \xi \in \mathbb{R}^{n_\xi} \mid \ell = 1,\ \tau - \tau_{min} \geq 0, \tau_{max} - \tau \geq 0\}$$

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Essential Idea

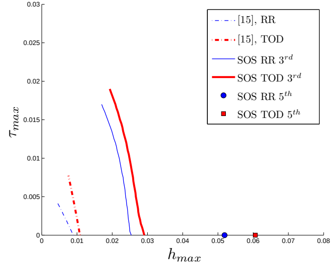
KEY STEPS:

1. NCS modeled as a hybrid system
2. LF candidate is structured polynomially $\rightarrow$ Flexibility
3. SOS + Hybrid Stability Theorem $\rightarrow$ LMI conditions for GAS
4. Embedded protocol in D $\rightarrow$ NCS is a piecewise poly hybrid system
5. regional info C_i, D_m $\rightarrow$ positivstellensatz (type of S-procedure)

$\rightarrow$ LMI conditions to prove GAS of the NCS!

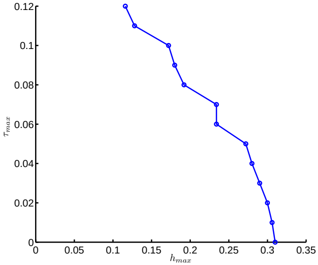
Numerical Examples

Batch Reactor



- 4 state plant
- 2 state controller
- y transmitted over network
- u wired
- $\tau_{min} = h_{min} = 0$

Polynomial Dynamics



$$\dot{x}(t) = -x^3(t) + x^2(t)u(t)$$
$$u(t) = -x(t)$$

- no comm. constraints

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Conclusions

Advantages

- automated method for stability analysis of NCSs
- less conservative than earlier methods
- piecewise polynomial plants and controllers
- non-zero lower bounds on varying delays and transmission intervals
- does not require an overapproximation of the NCS

Disadvantages

- computation complexity grows rapidly with state dimension
  $\rightarrow$ Improved solvers needed

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